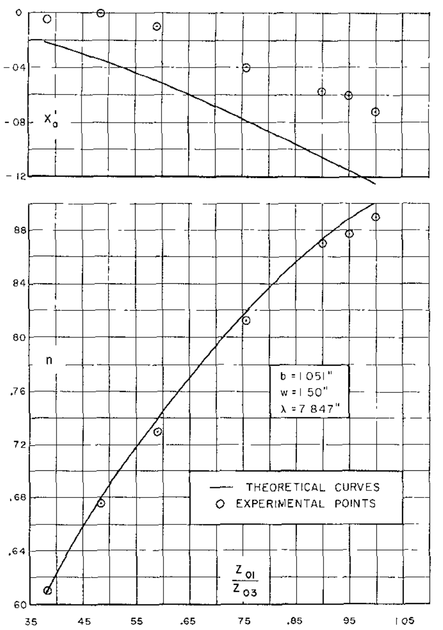

Fig. 23—Two of the circuit parameters for the symmetric tee junction.

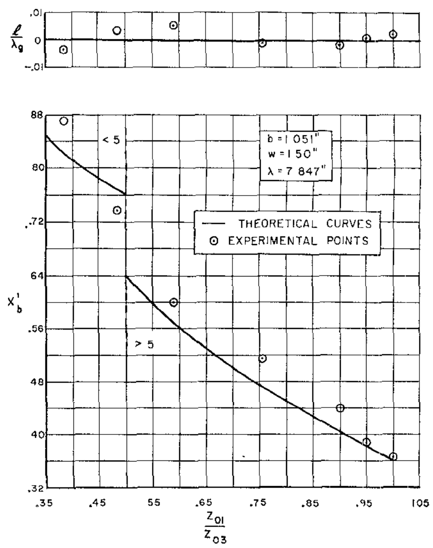

Fig. 24—The remaining two circuit parameters for the symmetric tee junction.

plicated structure in the presence of more significant parameters. The agreement for X_b' is actually much better (5 per cent or less) than a first glance at the curve would indicate. The two sections of the X_b' curve were computed by the formulas appropriate to the respective ranges. The experimentally determined length of line l (which is presented in Fig. 24 as normalized to guide wavelength) is seen to be sufficiently small to confirm the corresponding theory, *i.e.*, the terminals of the transformer are located so close to T_3 that line l is not required in the representation.

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A Variational Integral for Propagation Constant of Lossy Transmission Lines*

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Summary—**By assuming that the current on a lossy transmission line flows in the axial direction, only a variational integral for the propagation constant can be readily obtained. This variational integral shows that the usual power loss method of evaluating the attenuation constant is valid for general transmission lines. This variational integral also shows that the perturbation of the loss-free phase constant is due to the increase in magnetic field energy caused by penetration of the field into the conductors.**

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Introduction

The dominant mode of propagation on a loss-free transmission line is a TEM wave. In the transverse plane both the electric field and magnetic field may be derived from the gradients of suitable scalar functions of the transverse coordinates. The current flows entirely in the axial direction. Practical lines have finite conductivity and hence finite losses. As a consequence, there must be a component of the Poynting vector directed into the conductors and this in turn implies at least a longitudinal component of electric field. In general, longitudinal components of both electric and magnetic fields will exist (some exceptions are the coaxial line, single wire line, and the infinitely wide parallel plate line). A longitudinal magnetic field will have associated with it transverse currents on the conductors. Since these transverse currents arise only because of the perturbation of the TEM mode into a mode with axial field components, they are small in magnitude compared with the axial current. Thus, the losses (proportional to current density squared) associated with the transverse currents will also be small compared with the losses due to the axial current. For a first approximation, the transverse currents may be neglected. When this is done, a variational integral for the propagation constant of a general lossy transmission line may be readily obtained.

The usual power-loss method of evaluating the attenuation constant of a lossy transmission line is based on an evaluation of the Joule heating loss in the conductors by assuming a current distribution identical with that for the loss-free line.[1] If P_L is the power loss per meter computed on this basis and P is the power flow along the line, the attenuation constant α is

$$\alpha = P_L/2P. \tag{1}$$

The variational integral to be presented will provide a justification of this method for general lossy transmission lines.

A Variational Formulation

For simplicity, a general two-conductor transmission line as in Fig. 1 will be considered although the

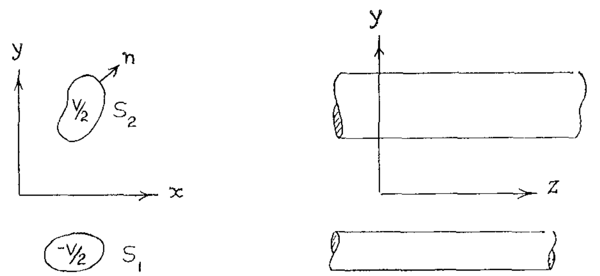

Fig. 1—A general two-conductor transmission line.

analysis is readily extended to multiconductor lines. If the current is assumed to be entirely in the axial direction and the medium surrounding the conductors to be homogeneous and isotropic, the field may be derived from a vector potential $\boldsymbol{A}$ having a z component $\psi(x, y)e^{-\gamma z}$ only. The relevant equations are

$$\boldsymbol{B} = \nabla x \boldsymbol{A} \tag{2a}$$

$$\boldsymbol{E} = -j\omega \boldsymbol{A} + \frac{\nabla\nabla\cdot\boldsymbol{A}}{j\omega\epsilon\mu_0} \tag{2b}$$

$$\nabla^2 \boldsymbol{A} + k^2\boldsymbol{A} = 0 \tag{2c}$$

[1] S. Ramo and J. R. Whinnery, "Fields and Waves in Modern Radio," John Wiley and Sons, Inc., New York, N. Y., 2nd ed., sec. 8.05; 1953.

where $k^2 = \omega^2\mu_0\epsilon$. For the present ϵ is assumed real. For a lossy dielectric medium surrounding the conductors ϵ is complex and shunt conductance losses are introduced. This requires only a trivial modification of the analysis, *i.e.*, replacing a real ϵ by a complex ϵ.

From (2b) and letting $\boldsymbol{A} = \boldsymbol{a}_z\psi e^{-\gamma z}$, it is found that

$$E_z = \frac{k_c^2}{j\omega\mu_0\epsilon}\psi e^{-\gamma z} \tag{3}$$

where $k_c^2 = k^2 + \gamma^2$. Provided the radius of curvature of the conductors S_1 and S_2 is much greater everywhere than the skin depth δ, the conductors will exhibit a surface impedance Z_m given by

$$Z_m = R_m + jX_m = (1 + j)/\sigma\delta \tag{4}$$

where σ is the conductivity and the skin depth δ is given by

$$\delta = (2/\omega\mu_0\sigma)^{1/2} \tag{5}$$

for a nonferrous material. The axial current density $\boldsymbol{J}$ is given by

$$\boldsymbol{J} = \boldsymbol{n}x\boldsymbol{H} = \mu_0^{-1}\boldsymbol{n}x(\nabla x\boldsymbol{A})$$
$$= \mu_0^{-1}[\nabla(\boldsymbol{n}\cdot\boldsymbol{A}) - (\boldsymbol{n}\cdot\nabla)\boldsymbol{A}] = -\mu_0^{-1}\frac{\partial \boldsymbol{A}}{\partial n} \tag{6}$$

where $\boldsymbol{n}$ is the unit outward normal from the conductors and $\boldsymbol{n}\cdot\boldsymbol{A} = 0$ since $\boldsymbol{n}$ and $\boldsymbol{A}$ are perpendicular. At the conductor surface $E_z = JZ_m$ and hence from (3) and (6) the boundary conditions for ψ are found to be

$$f\frac{\partial\psi}{\partial n} + \psi = 0 \text{ on } S_1, S_2 \tag{7}$$

where, for convenience, $j\omega\epsilon Z_m/k_c^2$ is denoted by f.

Let $\boldsymbol{E}_t e^{-\gamma z}$ be the transverse electric field. From (2b)

$$\boldsymbol{E}_t = \frac{j\omega\gamma}{k^2}\nabla\psi. \tag{8}$$

If there are no transverse currents, $\boldsymbol{n}x\boldsymbol{E}_t$ or $\boldsymbol{n}x\nabla\psi$ must vanish on S_1, S_2. This condition is in general incompatible with (7) except for the loss-free transmission line or lines with a high degree of symmetry, *e.g.*, coaxial line. For later use the solution for the loss-free line will be outlined. The vector potential z component will be taken as ψ_0. The propagation constant γ is equal to jk and $k_c = 0$. If the potentials of S_2 and S_1 are $V/2$ and $-V/2$ respectively, then

$$\psi_0 = (\mu_0\epsilon)^{1/2}V/2 \text{ on } S_2$$
$$= -(\mu_0\epsilon)^{1/2}V/2 \text{ on } S_1. \tag{9}$$

For the general case a variational integral for k_c^2 may be derived as follows. The equation satisfied by ψ is

$$\nabla_t^2\psi + k_c^2\psi = 0 \tag{10}$$

where ∇_t is the transverse operator $\mathbf{a}_x(\partial/\partial x)+\mathbf{a}_y(\partial/\partial y)$. Multiply (10) by ϕ, as yet an arbitrary function of x, y, that is regular at infinity, and integrate over the whole xy plane to get

$$k_c^2 \iint \phi\psi da = -\iint \phi\nabla_t^2\psi da.$$

Using Green's second theorem this may be rewritten as

$$k_c^2 \iint \phi\psi da = -\iint \psi\nabla_t^2\phi da + \oint_C \phi\frac{\partial\psi}{\partial n} dl - \oint_C \psi\frac{\partial\phi}{\partial n} dl \tag{11}$$

where n is the inward normal to the contour C. The contour C consists of the boundaries C_1, C_2 of the two conductors, the circle C_0 at infinity, and suitable cuts to make the region under consideration simply connected as in Fig. 2. Using the boundary condition (7) in (11)

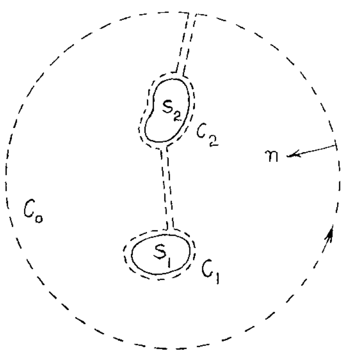

Fig. 2—Illustration of closed contour $C=C_0+C_1+C_2$.

gives

$$k_c^2 \iint \phi\psi da = -\iint \psi\nabla_t^2\phi da + \oint_{C_1+C_2}\left(\phi\frac{\partial\psi}{\partial n} + f\frac{\partial\psi}{\partial n}\frac{\partial\phi}{\partial n}\right) dl \tag{12}$$

since the contour integral around C_0 vanishes because ψ and ϕ are both regular at infinity. The first variation of (12) is now computed by standard methods to yield[2]

$$2k_c\delta k_c \iint \phi\psi da = -k_c^2\iint \psi\delta\phi da$$
$$-\iint \psi\nabla_t^2\delta\phi da - \iint \delta\psi(k_c^2\phi + \nabla_t^2\phi) da$$
$$+\oint_{C_1+C_2}\left(\phi + f\frac{\partial\phi}{\partial n}\right)\frac{\partial\delta\psi}{\partial n} dl + \oint_{C_1+C_2}\left(\delta\phi + f\frac{\partial\delta\phi}{\partial n}\right)\frac{\partial\psi}{\partial n} dl.$$

[2] P. M. Morse and H. Feshback, "Methods of Theoretical Physics," McGraw-Hill Book Co., Inc., New York, N. Y., vol. 2, sec. 9.4; 1953.

The second term on the right may be transformed using Green's second theorem, *i.e.*,

$$-\iint \psi\nabla_t^2\delta\phi da = -\iint \delta\phi\nabla_t^2\psi da + \oint_{C_1+C_2}\left(\psi\frac{\partial\delta\phi}{\partial n} - \delta\phi\frac{\partial\psi}{\partial n}\right) dl$$

and hence, the variation in k_c is given by

$$2k_c\delta k_c \iint \phi\psi da = -\iint \delta\psi(\nabla_t^2\phi + k_c^2\phi) da$$
$$-\iint \delta\phi(\nabla_t^2\psi + k_c^2\psi) da$$
$$+\oint_{C_1+C_2}\left(\phi + f\frac{\partial\phi}{\partial n}\right)\frac{\partial\delta\psi}{\partial n} dl$$
$$+\oint_{C_1+C_2}\left(\psi + f\frac{\partial\psi}{\partial n}\right)\frac{\partial\delta\phi}{\partial n} dl. \tag{13}$$

An examination of this result shows that the variation in k_c vanishes provided both ψ and ϕ satisfy the scalar Helmholtz equation (10) and the boundary condition (7). It now follows that if ϕ is replaced by ψ in (12) the resulting equation is a variational expression for k_c^2. Thus[3]

$$k_c^2 = \frac{-\iint \psi\nabla_t^2\psi da + \oint_{C_1+C_2}\left(f\frac{\partial\psi}{\partial n} + \psi\right)\frac{\partial\psi}{\partial n} dl}{\iint \psi^2 da}. \tag{14}$$

Substitution of a first-order approximate solution for ψ into (14) yields a solution for k_c^2 correct to the second order. For low-loss lines a suitable trial function to use in (14) is the corresponding solution ψ_0 for the loss-free line. The integral in the denominator serves as a normalization integral only. Once k_c^2 has been found, the propagation constant γ may be obtained at once from the relation $\gamma=(k_c^2-k^2)^{1/2}$.

Evaluation of k_c^2

The true potential function ψ_0 for the loss-free transmission line will be used as a trial function in (14). This function is a solution of Laplace's equation in the xy plane and hence $\nabla_t^2\psi_0=0$. Therefore, only the contour integral in (14) needs to be evaluated. Furthermore, ψ_0 will be normalized so that

$$\iint \psi_0^2 da = \mu_0\epsilon. \tag{15}$$

[3] This is not the only possible variational expression for k_c^2. However, this particular choice is a convenient one from which to obtain approximate results.

On S_1 and S_2 the current density J is equal to $-\mu_0^{-1}(\partial\psi/\partial n)$ with $\partial\psi/\partial n$ being negative on S_2 and positive on S_1. Also on S_1, $\psi_0=-(\mu_0\epsilon)^{1/2}\, V/2$, and on S_2, $\psi_0=(\mu_0\epsilon)^{1/2}\, V/2$. Substituting into (14) gives

$$k_c^2 = (\mu_0\epsilon)^{-1}\oint_{C_1+C_2}\left[\frac{j\omega\epsilon\mu_0 Z_m}{k_c^2}\mu_0 J^2 - \mu_0\psi J\right]dl$$
$$= \oint_{C_1+C_2}\frac{j\omega Z_m}{k_c^2}\mu_0 J^2 dl - Z_0\frac{V}{2}\oint_{C_2} J dl + Z_0\frac{V}{2}\oint_{C_1} J dl \qquad (16)$$

where $Z_0=(\mu_0/\epsilon)^{1/2}$. On S_1 the current is oppositely directed to that on S_2 so the last two integrals are together equal to $-Z_0VI=-2Z_0P$ where V is the potential difference between S_1 and S_2 and I is the total current flowing on one conductor and P is the power flow along the line. In the first integral on the right-hand side,

$$R_m\oint_{C_1+C_2} J^2 dl = 2P_L, \qquad (17)$$

where P_L is the conductor loss per meter. Since the conductivity is finite, the magnetic field penetrates into the conductor and a net amount of magnetic energy is stored in this internal field. At the surface, the magnetic field is equal to the current density J in magnitude and decays exponentially with distance u into the conductor according to $e^{-u/\delta}$. Hence, the internal magnetic energy is

$$W_{mi} = \frac{1}{4}\mu_0\int_0^{u_0}\oint_{C_1+C_2} J^2 e^{-2u/\delta} du dl$$

where the integral over u need be taken only to some interior point u_0 where the field is negligible. Integrating over u gives

$$W_{mi} = \frac{1}{8}\mu_0\delta\oint_{C_1+C_2} J^2 dl = P_L/2\omega. \qquad (18)$$

An internal inductance per meter may be defined by the relation

$$\tfrac{1}{4}I^2 L_i = W_{mi}$$

and from (18)

$$\omega L_i = R = 2P_L/I^2 \qquad (19)$$

where R is the equivalent series resistance of the line per meter. Using these results the first integral in (16) becomes $(2kZ_0/k_c^2)(jP_L-2\omega W_{mi})$. Finally, the equation for k_c^2 becomes

$$k_c^4 + 2Z_0Pk_c^2 - 2kZ_0(jP_L - 2\omega W_{mi}) = 0. \qquad (20)$$

The solution for k_c^2 is

$$k_c^2 = -Z_0P + [(Z_0P)^2 + 2kZ_0(jP_L - 2\omega W_{mi})]^{1/2}$$
$$\approx \frac{k}{P}(jP_L - 2\omega W_{mi}) \qquad (21)$$

where the binomial expansion could be used since P_L and W_{mi} are small compared with P for low-loss lines. Replacing k_c^2 by $k^2+(j\beta+\alpha)^2$ gives

$$2\beta\alpha = \frac{kP_L}{P} \qquad (22a)$$

$$k^2 - \beta^2 + \alpha^2 = -\frac{2W_{mi}}{P}\omega k. \qquad (22b)$$

Since the losses are assumed small $\beta\approx k$ and $k^2-\beta^2$ may be replaced by $2k(k-\beta)$. The right-hand side of (22b) is of magnitude $2\alpha k$ and, hence, α^2 is negligible in comparison. Therefore, (22) reduces essentially to the more familiar expressions

$$\alpha = P_L/2P \qquad (23a)$$

$$\beta = k + \omega W_{mi}/P. \qquad (23b)$$

For a lossless line $k=\omega\sqrt{LC}$ where L and C are the inductance and capacitance per unit length. Also, the characteristic impedance Z_c is equal to $(L/C)^{1/2}$ and

$$P_L = \tfrac{1}{2}RI^2, \qquad P = \tfrac{1}{2}Z_cI^2.$$

Therefore, (23) may be rewritten as follows:

$$\alpha = R/2Z_c \qquad (24a)$$

$$\beta = \omega[C(L + L_i)]^{1/2}. \qquad (24b)$$

The phase constant β is increased by an amount corresponding to the increase in the inductance of the line per meter due to addition of internal inductance. Since ωL_i and R are introduced together because of the finite conductivity, the series resistance of the line may be considered as equivalent to making the permeability of the medium surrounding the conductors complex. This provides a formal analogy between series resistance loss and shunt conductance loss as follows:

Lossless Line	*Lossy Line*
$\epsilon = \epsilon$	$\epsilon = \epsilon[1 - jG/\omega C]$
$\mu = \mu_0$	$\mu = \mu_0\left[1 + \frac{L_i}{L} - \frac{jR}{\omega L}\right]$

where G is the shunt conductance, L_i the internal inductance, L the external inductance, and R the series resistance of the line.

Conclusion

The use of a variational integral for k_c^2 verifies the validity of the usual power-loss method for evaluating the attenuation constant of low-loss transmission lines. In addition, it provides a method whereby k_c^2 can be evaluated to any desired degree of accuracy by using a trial function containing several variational parameters and determining these so as to make (14) stationary. The method is, however, limited to those cases where the transverse currents can be neglected in comparison with the axial currents.